$f(t)$ HAS SUPPORT $\left[-\frac{T}{2}, \frac{T}{2}\right]$

$$g(t) = \begin{cases} f(t), & -\frac{T}{2} \le t \le \frac{T}{2} \\ \text{PERIODIC W/ PERIOD } T \end{cases}$$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\left(\frac{2\pi}{T}\right)t}$$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-in\left(\frac{2\pi}{T}\right)t} \, dt$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\left(\frac{2\pi}{T}\right)t} \, dt$$

$$c_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot \frac{e^{-i\left(\frac{2\pi}{T}\right)nt}}{T} \, dt$$

LET $w = \frac{2\pi n}{T}$

$$C_{\frac{wT}{2\pi}} = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-iwt} \cdot \frac{dt}{T}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\omega t}\,dt$$

$$\hat{f}\left(\frac{2\pi n}{T}\right) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\frac{2\pi n}{T}t}\,dt$$

$$= T \cdot C_n \qquad \Rightarrow \qquad \boxed{C_n = \frac{\hat{f}\left(\frac{2\pi n}{T}\right)}{T}}$$

$$\boxed{\hat{f}\left(\frac{2\pi n}{T}\right) = T \cdot C_n}$$

$$\int_{-\infty}^{\infty} h(t)\,dt = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} h(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot T \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} h(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} T \cdot h(t)\,dt$$

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\left(\frac{2\pi n}{T}\right)t}\,dt$$

$u = f(t)$

$du = f'(t)\,dt$

$dv = e^{-i\left(\frac{2\pi n}{T}\right)t}\,dt$

$v = \frac{iT}{2\pi n}e^{-i\left(\frac{2\pi n}{T}\right)t}$

$$= \frac{1}{T}\left[\frac{iT}{2\pi n}f(t)\Big|_{-\frac{T}{2}}^{\frac{T}{2}} - \frac{iT}{2\pi n}\int_{-\frac{T}{2}}^{\frac{T}{2}} f'(t)e^{-i\left(\frac{2\pi n}{T}\right)}\,dt\right]$$

$$= \frac{-i}{2\pi n}\left[f(t)\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \int_{-\frac{T}{2}}^{\frac{T}{2}} f'(t)e^{-i\left(\frac{2\pi n}{T}\right)}\,dt\right]$$

AS $T \to \infty$

$$\to -\frac{i}{2\pi n}\int_{-\infty}^{\infty} f'(t)\, e^{-i\omega t}\, dt \qquad \text{WHERE } \omega = \frac{2\pi n}{T}$$

$$u = e^{-i\omega t} \qquad dv = f'(t)\,dt$$

$$du = -i\omega e^{-i\omega t}\, dt \qquad v = f(t)$$

$$\to \frac{-i}{2\pi n}\left[\overbrace{f(t)\, e^{-i\omega t}\Big|_{-\infty}^{\infty}}^{0} + i\omega \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \right]$$

$$= \frac{\omega}{2\pi n} \underbrace{\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt}_{\hat{f}(\omega)}$$

As $T \to \infty$,

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\left(\frac{2\pi n}{T}\right)t}\, dt$$

$$\omega = \frac{2\pi n}{T}$$

$$\frac{\omega T}{2\pi}$$

$$\to \frac{\omega}{2\pi n} \hat{f}(\omega)$$

$$C_{\frac{\omega T}{2\pi}} \to \frac{\omega}{2\pi n} \hat{f}(\omega)$$

DEFINE
$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} \, dt$$

SUPPOSE $f(t)$ HAS SUPPORT CONTAINED IN $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

THEN,
$$\hat{f}(w) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-iwt} \, dt$$

$$\hat{f}\left(\frac{2\pi n}{T}\right) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\left(\frac{2\pi n}{T}\right)t} \, dt = T \cdot c_n$$

So,

$$\boxed{c_n = \frac{1}{T}\hat{f}\left(\frac{2\pi n}{T}\right)}$$

For $w = \frac{2\pi n}{T} \Rightarrow n = \frac{wT}{2\pi}$

$$\boxed{\hat{f}(w) = T \cdot c_{\frac{wT}{2\pi}}}$$

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\left(\frac{2\pi n}{T}\right)t}\,dt$$

LET $w = \frac{2\pi n}{T}$

THEN $n = \frac{wT}{2\pi}$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-iwt}\,dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-iwt}\,dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} \tilde{f}(t)e^{-iwt}\,dt = \frac{1}{T}\hat{\tilde{f}}(w)$$

$$= \frac{1}{T}\hat{\tilde{f}}\left(\frac{2\pi n}{T}\right)$$

$$\int_{-\infty}^{\infty} \hat{f}(w)\, e^{i2\pi wt}\, dw = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt \right] dw$$

$$= \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} e^{-iwt}\, dw \right] dt$$

$$\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega = \int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} \left[\cos(\omega t) + i\sin(\omega t)\right] d\omega$$

$$= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\cos(\omega t) + i\sin(\omega t)\right] d\omega$$

$$= 2\lim_{T\to\infty} \int_{0}^{\frac{T}{2}} \cos(\omega t)\, d\omega$$

$$= 2\lim_{T\to\infty} \left.\frac{\sin(\omega t)}{t}\right|_{\omega=0}^{\omega=\frac{T}{2}}$$

$$= 2\lim_{T\to\infty} \left[\frac{\sin\left(\frac{T}{2}t\right)}{t}\right] = \lim_{T\to\infty} \left[\frac{\sin\left(\frac{t}{2}\cdot T\right)}{\frac{t}{2}}\right]$$

$$\lim_{T\to\infty}\left[\frac{\sin\left(\frac{t}{2}\cdot T\right)}{\frac{t}{2}}\right] = 0 \text{ IF } t \neq 0$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega\right] dt = \int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{\sin\left(\frac{t}{2}\cdot T\right)}{\frac{t}{2}}\right] dt$$

$$= \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{\sin\left(\frac{T}{2}\cdot t\right)}{\frac{t}{2}}\, dt$$

$$u = \frac{T}{2}t \Rightarrow \frac{u}{T} = \frac{t}{2}$$

$$du = \frac{T}{2}dt \Rightarrow \frac{2}{T}\, du = dt$$

$$= \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{\sin u}{\frac{u}{T}}\cdot\frac{2}{T}\, du = \lim_{T\to\infty} \cdot 2\int_{-\infty}^{\infty}\frac{\sin u}{u}\, du$$

$$= \lim_{T\to\infty} 2\pi = 2\pi$$

SO,

$$\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega = 0 \quad \text{IF} \quad t \neq 0$$

AND

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega \right] dt = 2\pi$$

SO,

$$\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega = 2\pi\, \delta(t)$$

$$\int_{-\infty}^{\infty} \hat{f}(w) e^{iwt}\, dw = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-iw\tau}\, d\tau \right] e^{iwt}\, dw$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} e^{iw(t-\tau)}\, dw \right] d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot 2\pi\, \delta(t-\tau)$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot 2\pi\, \delta(\tau - t)$$

$$= 2\pi f(t)$$

THUS, IF

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

THEN,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega$$

(1) $c_n = \frac{1}{T}\hat{\bar{f}}\left(\frac{2\pi n}{T}\right)$

WHERE $\bar{f}(t) = f(t)\,\chi_{[-\frac{T}{2},\frac{T}{2}]}(t)$

AND $\hat{g}(w) = \int_{-\infty}^{\infty} g(t)\,e^{-iwt}\,dt$

(2) $\int_{-\infty}^{\infty} \frac{\sin u}{u}\,du = 2\int_{0}^{\infty} \frac{\sin u}{u}\,du = 2\cdot\frac{\pi}{2} = \pi$

(3) $\int_{-\infty}^{\infty} e^{iwt}\,dw = 2\pi\,\delta(t)$

(4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)\,e^{iwt}\,dw = f(t)$

$$I(v) = \int_0^\infty \frac{\sin u}{u} \cdot e^{-vu} \, du$$

$$I'(v) = \int_0^\infty \frac{\sin u}{u} \cdot (-u) e^{-vu} \, du$$

$$= -\int_0^\infty \sin u \cdot e^{-vu} \, du$$

$$= \left. \frac{(v \sin u + \cos u) e^{-vu}}{v^2+1} \right|_{u=0}^{u=\infty}$$

$$= -\frac{1}{v^2+1} \qquad \Rightarrow \quad I(v) = -\tan^{-1}(v) + C$$

SO, $\int_0^\infty \frac{\sin u}{u} \cdot e^{-vu}\, du = -\tan(v) + C$

AS $v \to \infty$, $\int_0^\infty \frac{\sin u}{u} \cdot e^{-vu}\, du \longrightarrow 0$

ALSO, $-\tan(v) + C \longrightarrow -\frac{\pi}{2} + C$

SO, $-\frac{\pi}{2} + C = 0 \Rightarrow C = \frac{\pi}{2}$

SO, $I(v) = \int_0^\infty \frac{\sin u}{u} \cdot e^{-vu}\, du = -\tan^{-1}(v) + \frac{\pi}{2}$

THUS, $I(0) = \boxed{\int_0^\infty \frac{\sin u}{u}\, du = \frac{\pi}{2}}$

$$J = \int \sin u \cdot e^{-vu}\, du = -\frac{\sin u\, e^{-vu}}{v} + \frac{1}{v}\int \cos u\, e^{-vu}\, du$$

$f = \sin u \qquad g' = e^{-vu}\, du$

$f' = \cos u\, du \qquad g = -\frac{e^{-vu}}{v}$

$f = \cos u \qquad g' = e^{-vu}\, du$

$f' = -\sin u \qquad g = -\frac{e^{-vu}}{v}$

$$= -\frac{\sin u \cdot e^{-vu}}{v} - \frac{\cos u \cdot e^{-vu}}{v^2} - \frac{1}{v^2}\int \sin u \cdot e^{-vu}\, du$$

$$J\underbrace{\left(1+\frac{1}{v^2}\right)}_{\frac{v^2+1}{v^2}} = -\frac{e^{-vu}}{v^2}\left(v\sin u + \cos u\right)$$

$$J = \frac{-(v\sin u + \cos u)\, e^{-vu}}{v^2+1} + C$$