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TEACHER VERSION

MODELING THE SMOKING PROCESS OF SOUTHERN BARBECUE

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Abstract: We offer raw data collected from two thermometers used in the smoking process of Southern barbecue. One thermometer measures the temperature inside of the smoke chamber and the other measures the internal temperature of the meat. This data can be used to model and predict the amount of time required to smoke meats for barbecue, but it can also be used to justify the existence of and quantify the temperatures that constitute a "stall" when smoking large cuts of meat.

Keywords: Barbecue, Newton's law of cooling/warming, modeling, logistic function

Tags: first order, ordinary differential equations, non-linear regression

STATEMENT

Introduction

Barbecue is a staple in Southern cuisine, and there are many types of meats as well as preparation and cooking styles among barbecue aficionados. In most Southern states, pork is the meat of choice while Texans prefer beef. Whether pork or beef, the cuts of meat used in Southern barbecue are typically high in fat and connective tissues. As such, these cuts require great care to prepare and thus are usually the most inexpensive of all cuts of meat.

The two primary cuts of pork used in Southern barbecue are ribs and front shoulders. Being the smaller of the two, ribs are generally desired because of their preparation time compared to shoulders. Pork ribs used in Southern barbecue include both spare ribs and loin back (or "baby back") ribs. Spare ribs are generally larger and have more fat and connective tissue than loin back ribs which makes them desirable for most barbecue pitmasters.

Pork shoulders can be separated into two categories, namely the picnic ham and the Boston butt. Boston butts are typically the more desirable cuts from the shoulder since they have more marbled fat (which keeps the meat moist and flavorful). Boston butts are generally between 5-10 pounds each, and can take between 6-24 hours to cook depending on the target temperatures of both the meat and cooking device. Boston butts are the typical cuts used to make "pulled pork" barbecue.

The primary cut of beef used in Southern barbecue is brisket. The brisket is a cut from the lower chest of the cow and is extremely tough and fatty due to the fact that it supports about 60% of the body weight of standing and moving cattle. Untrimmed beef briskets typically weigh between 8-20 pounds (depending on the size of the cow), and they generally take the longest time to prepare and cook of the meats used in Southern barbecue.

Whether they are cooking pork ribs, Boston butts, or beef briskets, barbecue pitmasters usually have at least one common strategy – "low and slow". That is, in order to make these fatty and tough cuts of meat desirable, the fat and connective tissues must be rendered away. This is most easily done by cooking the meat slowly at a low temperature (usually between 225° and 250° degrees Fahrenheit). This "low and slow" process transforms fatty and tough cuts of meat into moist and tender culinary masterpieces.

Ribs, Boston butts, and beef briskets can all be cooked in conventional ovens and slow cookers. However, authentic Southern barbecue requires one ingredient that these devices cannot provide – smoke. True Southern barbecue infuses smoke into the meat while it is being cooked, and this is most easily done using barbecue "pits". The styles, sizes, and prices of barbecue pits are almost endless, but all pits have the common feature of providing smoke to the meat while it cooks. The smoke provided by these pits comes from burning wood, and there are varieties of hardwoods used by barbecue pitmasters which include, but are not limited to, hickory, pecan, oak, mesquite, apple, cherry, and peach. Each of these woods has a distinct flavor, and the types of wood used are typically determined by preference and availability.

Data Collection

In order to mathematically model the smoking process of large cuts of meat like Boston butts and briskets, we use the "Stoker" by Rock's BBQ [1]. Rock's BBQ generously donated this device to the author in 2008 for the purpose of scientific data collection, and Figure 1 illustrates the original (wired) version of the device.

The Stoker was designed to connect a blower to the device in order to stoke the fire when the temperature falls below a minimum value. It has an easy to use web interface, and it is also accessible using the Telnet protocol over Transmission Control Protocol (TCP). Using the Telnet protocol, the temperature of both the smoke chamber and the meat can be monitored (virtually) continuously with temperatures updated about every five seconds. The author has written Perl software to access

Figure 1. Rock's BBQ Stoker device with attached thermometers.

the temperature data over Telnet, record the data into a data file, and serve the raw and entire session graph of the data in real-time.

The cut of meat used in the collection process for this paper was a 10.7 pound beef brisket. Figures 2 through 5 illustrates the preparation, smoking, and final slicing of the brisket.

Figure 2 shows the untrimmed brisket immediately out of the packaging. A beef brisket has two components – a leaner part called the "flat" and a fattier part called the "point". Figure 3 shows the brisket after it has been trimmed and separated. Figure 4 shows both the beef brisket as well as a Boston butt after being placed in the smoker. Figures 5 and 6 illustrate the finished sliced barbecue brisket.

Assignment

The provided data file [2] contains three columns of tabular data. The first column is time t in hours, the second column is the smoke chamber temperature $h(t)$ in degrees Fahrenheit, and the third column is the internal meat temperature $f(t)$ in degrees Fahrenheit. Use the provided data file to

1. Graph $f(t)$ and $h(t)$ versus time t.
2. Identify the interval where the rate of change of $f(t)$ is small (i.e., the interval where the temperature of the meat stalls). This partitions the time duration into three intervals $[t_0, t_1)$, $[t_1, t_2)$, and $[t_2, t_3)$ called Stage I, Stage II, and Stage III, respectively where $t_0 = 0$ and $t_3 = 11.35$.
3. Explain why the following logistic model is a reasonable model for $f(t)$ on Stage I and Stage III.
$$\frac{dy}{dt} = \lambda(y - k)(K - y)$$
4. Verify that
$$y(t) = k + \frac{K - k}{1 + De^{-\lambda(K-k)(t-\gamma)}}$$
is the general solution to the following differential equation.
$$\frac{dy}{dt} = \lambda(y - k_1)(K - y)$$

Figure 2. Untrimmed Brisket.

Figure 3. Trimmed/Separated Brisket.

Figure 4. Brisket and Shoulder.

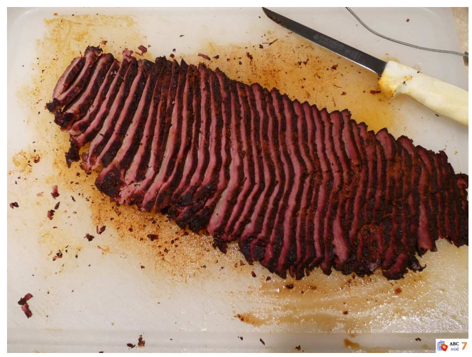

Figure 5. Sliced Brisket.

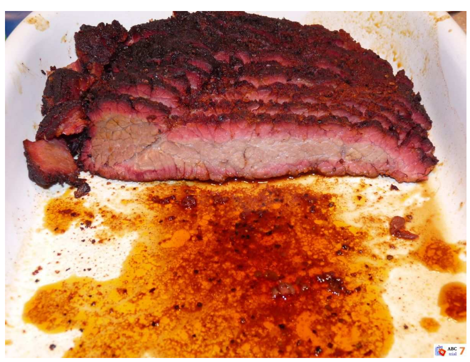

Figure 6. Finished Brisket.

5. Estimate the parameters k_1, K_1, D_1, λ_1, and γ_1 for which

$$g_1(t) = k_1 + \frac{K_1 - k_1}{1 + D_1 e^{-\lambda_1 (K_1 - k_1)(t - \gamma_1)}}$$

is a reasonable model for $f(t)$ on Stage I.

6. Estimate the parameters k_3, K_3, D_3, λ_3, and γ_3 for which

$$g_3(t) = k_3 + \frac{K_3 - k_3}{1 + D_3 e^{-\lambda_3 (K_3 - k_3)(t - \gamma_3)}}$$

is a reasonable model for $f(t)$ on Stage III.

7. Find a linear function $g_2(t) = m_2 t + b_2$ defined on Stage II such that

$$g(t) = \begin{cases} g_1(t), & t_0 \leq t < t_1 \\ g_2(t), & t_1 \leq t < t_2 \\ g_3(t), & t_2 \leq t < t_3 \end{cases}$$

is continuous on $[0, 11.35)$.

8. Graph $f(t)$ and $g(t)$ versus t.
9. Determine the root mean square error between $f(t)$ and $g(t)$ on $[0, 11.35)$.
10. Discuss whether or not you would expect the internal temperature of another 10.7 lb beef brisket to behave exactly like $f(t)$.

Conclusion

Smoking Southern barbecue is a time consuming process, and when executed correctly, transforms fatty and tough cuts of meat into moist and tender masterpieces. Not only is the finished product delectable, but the cooking process can be modelled mathematically to confirm and explain the existence of a stall (or plateau). There are several theories as to the cause of meat stalling when cooked slowly, and the stall induces a natural delineation of three stages throughout the cooking process – namely pre-stall, stall, and post-stall. This study has demonstrated that the pre-stall and post-stall stages can be modelled by logistic differential equations while the stall itself can be modelled linearly.

REFERENCES

[1] Rock's BBQ.2016. Rock's BBQ in Fremont, CA. `http://www.rocksbarbecue.com`.

[2] The data file, bbq.d, is available in the electronic version of this Modeling Scenario.

COMMENTS

Proposed Solution

The brisket used for the provided data set smoked for about 11 hours and 21 minutes, and Figure 7 is a graph of the smoke chamber and internal meat temperatures (in degrees Fahrenheit) over this period. Question 1 from the Assignment requires the student to graph the smoke chamber and internal meat temperatures versus time.

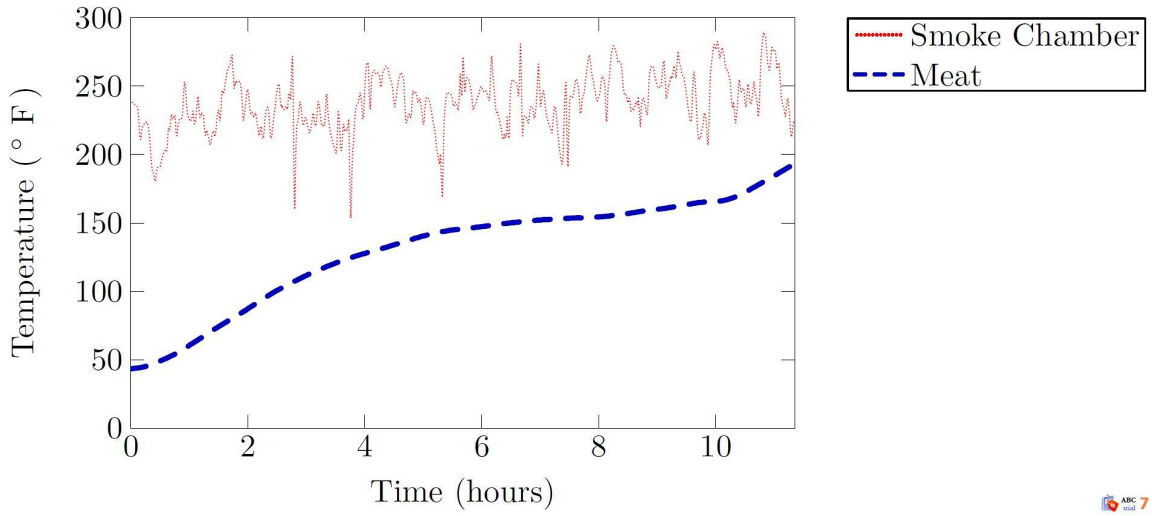

Figure 7. Smoke Chamber and Internal Meat Temperatures

Stall

Most all large cuts of meats enter a temperature "stall" (which we define below) when cooked at low temperatures. This is evident in the temperature data illustrated in Figure 7. The internal temperature of the meat increases at a much slower rate between about 6.6 and 10 hours. The internal meat temperature increases at a rate of about 4.5°F per hour whereas the temperature increases at a much faster rate prior to 6.6 hours and after 10 hours. Therefore, the first step in modelling the smoking process is quantifying a temperature "stall". Question 2 of the Assignment requires the student to determine a reasonable range for the stall.

If we let $f(t)$ be the internal temperature of the meat at any time t, then, $f'(t)$ is the rate of change of the meat temperature. Furthermore, if the magnitude of $f(t)$ is large, then the corresponding magnitude of $f'(t)$ may be large. Therefore, we define the exponential growth rate of $f(t)$ to be

$$\alpha(t) = \frac{f'(t)}{f(t)}.$$

Since the meat's initial temperature is about 43°F, and since it never falls below this temperature, $f(t)$ is, in particular, never 0; therefore, $\alpha(t)$ is well-defined for all $t \geq 0$.

From visual inspection of the plot of the data in Figure 7, it follows that $\alpha(6.6) \approx 0.03$ and $\alpha(10) \approx 0.03$. Furthermore, it follows that $|\alpha(t)| \leq 0.03$ for all $t \in (6.6, 10)$ and $|\alpha(t)| > 0.03$ for all $t \in (0, 6.6) \cup (10, 11.35)$. This gives rise to the following empirical definition of a "stall".

We say that the internal temperature of the meat has *stalled* whenever $|\alpha(t)| \leq 0.03$.

Figure 8 illustrates the three intervals and refers to each of these intervals as *stages* in the smoking process.

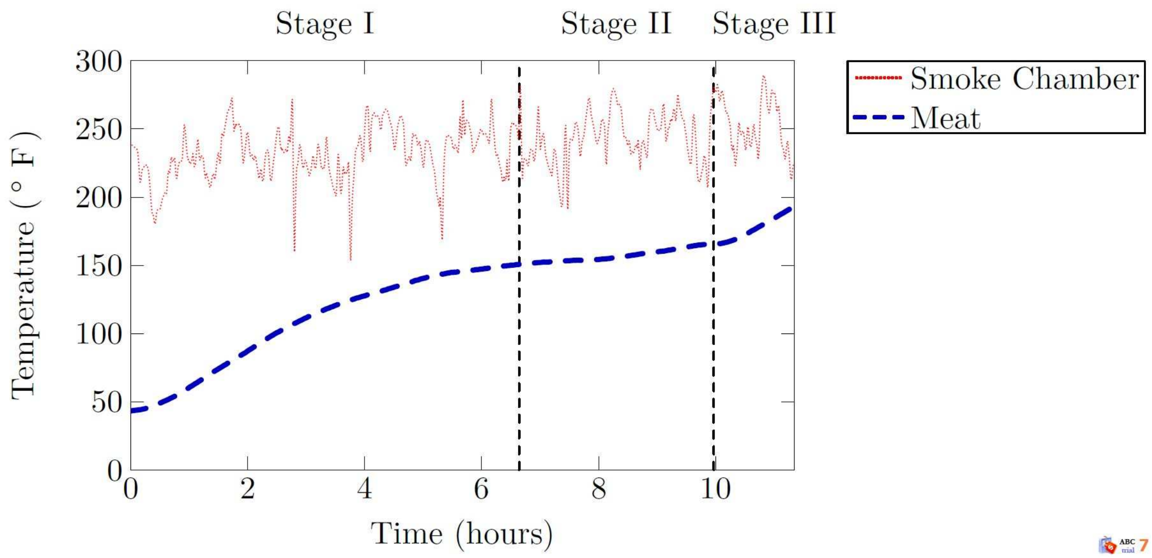

Figure 8. Stages of the Internal Meat Temperatures

If we define Stage I to be the period of time before the stall, Stage II to be the stall itself, and Stage III to be the period of time after the stall, then the average rate of change of $f(t)$ in Stage I, Stage II, and Stage III is 16.2°F, 4.5°F, and 19.9°F per hour, respectively. These quantities confirm the appropriateness of the definition of the stall.

Stage I

We now consider the problem of modelling $f(t)$ in each stage of the smoking process. Since $f(t)$ resembles an S-shaped curve in Stage I, a natural model for $f(t)$ in this stage is the logistic model $g_1(t)$ defined by

$$\begin{aligned} g_1(t) &= k_1 + \frac{K_1 - k_1}{1 + C_1 e^{-\lambda_3 (K_1 - k_1) t}} \\ &= k_1 + \frac{K_1 - k_1}{1 + D_1 e^{-\lambda_3 (K_1 - k_1)(t - \gamma_1)}} \end{aligned} \tag{1}$$

where k_1, K_1, λ_1, and C_1 are parameters for the model and $C_1 = D_1 e^{\lambda_3 (K_1 - k_1)\gamma_1}$. We use the form of $g_1(t)$ in (1) for numerical stability, and we use GNU Octave's Levenberg-Marquardt nonlinear regression function `leasqr` to minimize the sum of the square errors between the internal temperature $f(t)$ of the meat and the corresponding model $g_1(t)$. Octave's `leasqr` function provides the values $k_1 \approx 3.3$, $K_1 \approx 154.8$, $D_1 \approx 0.105$, $\lambda_1 \approx 0.0045$, and $\gamma_1 \approx 5$. Thus $K_1 - k_1 \approx 151.49$, and so $\lambda_1(K_1 - k_1) \approx 0.682$; therefore,

$$g_1(t) \approx 3.3 + \frac{151.49}{1 + 0.105 \cdot e^{-0.682(t-5)}}$$

and Question 5 of the Assignment requires the student to estimate the parameters k_1, K_1, D_1, λ_1, and γ_1.

Functions of the form of (1) satisfy the first-order logistic ordinary differential equation

$$\frac{dy}{dt} = \lambda\,(y-k)(K-y) \tag{2}$$

where k is the "lower capacity", K is the "upper capacity", and λ is the "growth rate" for the solution $y(t)$ of (2). Question 4 of the Assignment requires the student to show that the form of (1) is the general solution to the differential equation (2). There are many theories as to why large cuts of meat stall when cooked at low temperatures, but it is generally agreed that if these meats stall, then they do so around 155 degrees Fahrenheit. Therefore, the value of $K_1 \approx 154.8$ is reasonable for an "upper capacity" for $f(t)$. For this choice of parameters k_1, K_1, D_1, λ_1, and γ_1, the root mean square error between $f(t)$ and $g_1(t)$ in Stage I is approximately 0.9 degrees.

Stage II

In Stage II, the rate of change of the internal temperature of the meat appears to be fairly constant. Therefore, we assume that during the stall, $f(t)$ can be modeled by a linear function $g_2(t)$ of the form

$$g_2(t) = m_2\,t + b_2$$

where m_2 is the constant rate of change and b_2 is the additional parameter needed to fit $f(t)$. This form for $g_2(t)$ is the general solution to the most elementary differential equation

$$\frac{dy}{dt} = m$$

where m is a constant. We may use Octave's `leasqr` function (as in Stage I) to find m_2 and b_2, but instead we will fit $f(t)$ in Stage III and linearly *connect* the models $g_1(t)$ and $g_3(t)$ in Stages I and III, respectively. Question 7 of the Assignment requires the student to determine a linear model $g_2(t)$ for $f(t)$ in Stage II that ensures this *connection*.

Stage III

In Stage III, the rate of change of $f(t)$ becomes very small as t approaches 10 hours (from the right). Also, because of Newton's Law of Warming, we expect the rate of change of $f(t)$ to become very small as $f(t)$ approaches the (ambient) temperature of the smoke chamber. Thus, we expect $f(t)$ to resemble an S-shaped curve in Stage III. Analogous to Stage I, we choose $g_3(t)$ of the form

$$g_3(t) = k_3 + \frac{K_3 - k_3}{1 + D_3 e^{-\lambda_3 (K_3 - k_3)(t-\gamma_3)}}.$$

As in Stage I, we use Octave's `leasqr` to minimize the sum of the square error between the internal temperature $f(t)$ of the meat and the model $g_3(t)$. Octave's `leasqr` function provides the values $k_3 \approx 161$, $K_3 \approx 204$, $D_3 \approx 0.187$, $\lambda_3 \approx 0.0569$, and $\gamma_3 \approx 11.6$. Thus, $K_3 - k_3 \approx 43$, and so $\lambda_3(K_3 - k_3) \approx 2.45$; therefore

$$g_3(t) \approx \; 161.1 + \frac{42.9}{1 + 0.18 \cdot e^{-2.45(t-11.62)}}$$

and Question 6 of the Assignment requires the student to estimate the parameters k_3, K_3, D_3, λ_3, and γ_3.

For this choice of k_3, K_3, λ_3, D_3, and γ_3, the root mean square error between $f(t)$ and $g_3(t)$ in Stage III is approximately 0.23 degrees. Figure 9 illustrates how closely $g_1(t)$ and $g_3(t)$ model $f(t)$ in Stages I and III, respectively; and it also illustrates how well the linear *connection* of $g_1(t)$ to $g_3(t)$ models $f(t)$ in Stage II. Question 8 of the Assignment requires the student to graph $f(t)$ as well as $g_1(t)$, $g_2(t)$, and $g_3(t)$ on the same graph versus t. The root mean square error between $f(t)$ and $g(t)$ for all $0 \le t < 11.35$ is about 1.35 degrees where

$$g(t) = \begin{cases} g_1(t), & t_0 \le t < t_1 \\ g_2(t), & t_1 \le t < t_2 \\ g_3(t), & t_2 \le t < t_3 \end{cases}$$

and $g_2(t)$ is the line making $g(t)$ continuous on $[t_0, t_3)$. Question 9 of the Assignment requires the student to compute the root mean square error between $f(t)$ and $g(t)$.

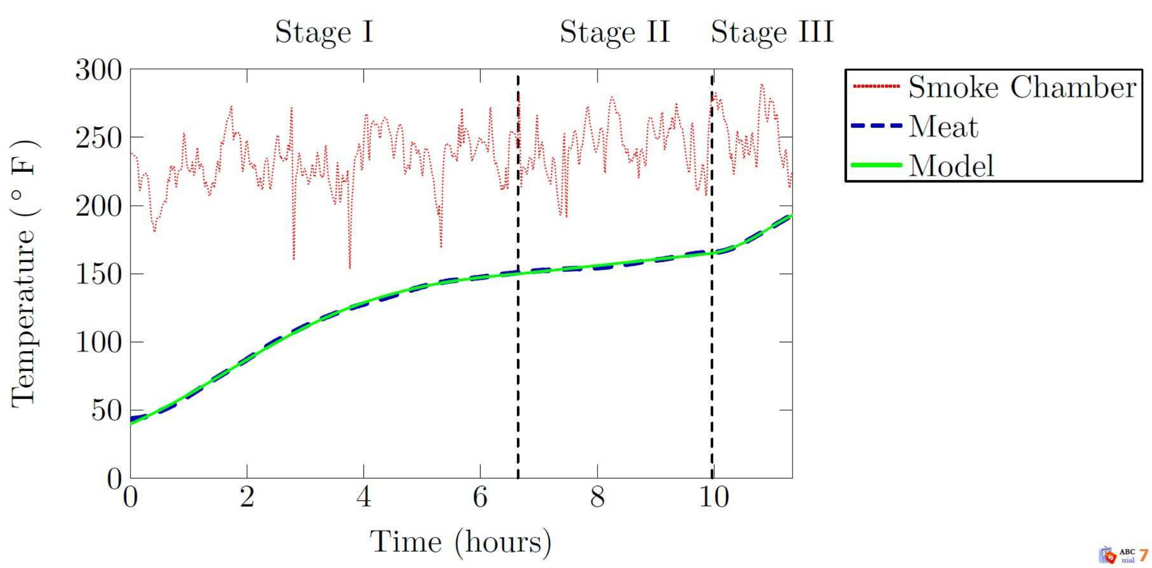

Figure 9. Temperature model fit in each stage