Quadratic Bézier curves

Let $P_0, P_1, P_2$ be distinct points. If points $U, V, B$ divide the line segments $P_0P_1, P_1P_2, UV$ by an equal ratio, then $B$ moves on a quadratic Bézier curve if $U$ moves about $P_0$. Thus for some real number $t$, where $0 \leq t \leq 1$.

\begin{align}
U - P_0 &= t \cdot (P_1 - P_0), \\
V - P_1 &= t \cdot (P_2 - P_1), \\
B - U &= t \cdot (V - U),
\end{align}

From the above equations, it follows that $U = P_0 + t \cdot (P_1 - P_0) = P_0 + t \cdot P_1 - t \cdot P_0$, therefore

$$U = (1 - t) \cdot P_0 + t \cdot P_1. \tag{1}$$

We equally find $V = (1 - t) \cdot P_1 + t \cdot P_2$ and $B = (1 - t) \cdot U + t \cdot V$. Substitution yields $B = (1 - t) \cdot (1 - t) \cdot P_0 + t \cdot P_1 + t \cdot (V = (1 - t) \cdot P_1 + t \cdot P_2)$, and after expansion,

$$B = (1 - t)^2 \cdot P_0 + 2(1 - t)t \cdot P_1 + t^2 \cdot P_2. \tag{2}$$

Cubic Bézier curves

Let $P_0, P_1, P_2, P_3$ be distinct points, and let the points $P_4, \ldots, P_8, B$ divide their respective line segments by an equal ratio:

\begin{align}
P_4 - P_0 &= t \cdot (P_1 - P_0), \\
P_5 - P_1 &= t \cdot (P_2 - P_1), \\
P_6 - P_2 &= t \cdot (P_3 - P_2), \\
P_7 - P_3 &= t \cdot (P_4 - P_3), \\
P_8 - P_4 &= t \cdot (P_5 - P_4), \\
B - P_7 &= t \cdot (P_8 - P_7),
\end{align}

where $0 \leq t \leq 1$. (In fig. 2 as well as in fig. 1, $t = 0.4$.)

From (2), we get

$$\begin{cases}
P_7 &= (1 - t)^2 \cdot P_0 + 2(1 - t)t \cdot P_1 + t^2 \cdot P_2, \\
P_8 &= (1 - t)^2 \cdot P_1 + 2(1 - t)t \cdot P_2 + t^2 \cdot P_3.
\end{cases}$$

From (1), $B = (1 - t) \cdot P_7 + t \cdot P_8 = (1 - t) \cdot [(1 - t)^2 \cdot P_0 + 2(1 - t)t \cdot P_1 + t^2 \cdot P_2] + t \cdot [(1 - t)^2 \cdot P_1 + 2(1 - t)t \cdot P_2 + t^2 \cdot P_3]$ which, after expansion, yields

$$B = (1 - t)^3 \cdot P_0 + 3(1 - t)^2t \cdot P_1 + 3(1 - t)t^2 \cdot P_2 + t^3 \cdot P_3. \tag{3}$$
In fig. 2, if \( P_4 \) moves about the segment \( P_0P_1 \), then \( P_7 \) moves on the quadratic Bézier curve determined by points \( P_0, P_1, P_2 \), while \( P_8 \) moves on the quadratic Bézier curve determined by points \( P_1, P_2, P_3 \).

**Bézier curves of arbitrary order**

For distinct points \( P_0, P_1, \ldots, P_n \), the Bézier curve of order \( n (n = 0, 1, 2, \ldots) \) can be recursively defined by

\[
B_0(t, P_0) \equiv P_0, \\
B_n(t, P_0, \ldots, P_n) \equiv (1 - t) \cdot B_{n-1}(t, P_0, \ldots, P_{n-1}) + t \cdot B_{n-1}(t, P_1, \ldots, P_n) \quad (n > 0),
\]

where \( 0 \leq t \leq 1 \).

**Theorem 1 (Bézier curves of order 1)** For points \( P_0, P_1 \), the Bézier curve of order 1 is given by the equation

\[
B_1(t, P_0, P_1) = (1 - t) \cdot P_0 + t \cdot P_1.
\]

**PROOF:** By the above definition, \( B_1(t, P_0, P_1) = (1 - t) \cdot B_0(t, P_0) + t \cdot B_0(t, P_1) = (1 - t) \cdot P_0 + t \cdot P_1 \). \( \square \)

**Theorem 2** For any non-negative integer \( n \), the Bézier curve of order \( n \) is given by the equation

\[
B_n(t, P_0, \ldots, P_n) = \sum_{k=0}^{n} \binom{n}{k} (1 - t)^{n-k} \cdot P_k.
\]

**PROOF:** By induction on \( n \). For \( n = 0 \), the theorem claims

\[
B_0(t, P_0) = \sum_{k=0}^{0} \binom{n}{k} (1 - t)^{n-k} = P_0 = P_0,
\]

which is correct by the first part of the definition.

For \( n > 0 \), we have \( B_n(t, P_0, \ldots, P_n) = (1 - t) \cdot B_{n-1}(t, P_0, \ldots, P_{n-1}) + t \cdot B_{n-1}(t, P_1, \ldots, P_n) \) by definition. By induction hypothesis,

\[
B_n(t, P_0, \ldots, P_n) = (1 - t) \sum_{k=0}^{n-1} \binom{n-1}{k} (1 - t)^{n-1-k} \cdot P_k + t \sum_{k=0}^{n-1} \binom{n-1}{k} (1 - t)^{n-1-k} \cdot P_{k+1}.
\]

We get

\[
(1 - t) \sum_{k=0}^{n-1} \binom{n-1}{k} (1 - t)^{n-1-k} \cdot P_k = (1 - t)^n \cdot P_0 + \sum_{k=1}^{n-1} \binom{n-1}{k} (1 - t)^{n-k} \cdot P_k,
\]

\[
t \sum_{k=0}^{n-2} \binom{n-1}{k} (1 - t)^{n-1-k} \cdot P_{k+1} = \sum_{k=0}^{n-2} \binom{n-1}{k} (1 - t)^{n-1-k+1} \cdot P_{k+1} + t^n \cdot P_n
\]

\[
= \sum_{k=1}^{n-1} \binom{n-1}{k} (1 - t)^{n-k} \cdot P_k + t^n \cdot P_n.
\]

Substituting the last two results, we get

\[
B_n(t, P_0, \ldots, P_n) = (1 - t)^n \cdot P_0 + \sum_{k=1}^{n-1} \left[ \binom{n-1}{k} + \binom{n-1}{k-1} \right] (1 - t)^{n-k} \cdot P_k + t^n \cdot P_n
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (1 - t)^{n-k} \cdot P_k, \quad \text{as} \quad \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}
\]

by the fundamental law of the binomial coefficients. \( \square \)